Magnetogasdynamic Channel Flows

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The asymptotic behavior of small unsteady perturbations that are superposed on an equilibrium magnetogasdynamic channel flow is considered. Only those channels with slowly varying cross-sectional area and small characteristic width-to-length ratio are allowed, so that variations along the channel are typically much smaller than are those across the channel. Under these conditions and the provision that the current density inside the duct is small, a set of "quasi-one-dimensional" equations is obtained. From the resulting set of hyperbolic equations, the theory of characteristics provides certain compatibility relations that describe the behavior of small disturbances of the steady channel flow, which are caused by some local pressure fluctuation. Two distinct wave motions are present within the channel, and the asymptotic behaviors of both are examined with the use of the aforementioned compatibility relations. It is shown that, under certain conditions on the magnetic-field distribution, wave motions that lead to shock formation in ordinary gasdynamic flow will attenuate because of the action of the magnetic field. Therefore, an instability is no longer predicted, and the possibility of a stable or shockless flow, due to the action of a suitably placed magnetic field, is revealed.

Nomenclature

a	$= (\partial p/\partial \rho)_\eta^{1/2}$, ordinary speed of sound
A	$= d \ln Q/dx$
$\mathbf{B}$	$= (B_x, B_y, 0)$, magnetic induction vector
c	$= (a^2 + \alpha^2)^{1/2}$, effective sound speed
d	= channel height at entrance
$\mathbf{E}$	$= (E_x, E_y, E_z)$, electric field vector
$\mathbf{H}$	$= (H_x, H_y, 0)$, magnetic field intensity
$\mathbf{h}$	$= (h_x, h_y, 0)$, induced magnetic field intensity
j	= electric current density in direction of z coordinate
j'	= perturbation electric current density
l	= characteristic length of channel
M_a	$= u/a$, Mach number based on ordinary sound speed
M_c	$= u/c$, Mach number based on effective sound speed
p	= gas pressure
P	= equilibrium gas pressure
$\mathbf{q}$	$= (u, v, 0)$, velocity vector
$Q(x)$	= channel height
R	= equilibrium gas density
R_m	$= 4\pi\mu\sigma U_\infty d$, magnetic Reynolds number
s	= stability parameter
S	$= \mu H_y^2/4\pi\rho U^2$, interaction parameter
t	= time, sec
T	= absolute temperature, °K
U	= equilibrium velocity in x direction
x, y, z	= Cartesian coordinates, defined in Fig. 1
z	$= 1/s$, stability parameter
α	$= H_y/(4\pi\rho/\mu)^{1/2}$, Alfvén velocity
β	$= a/c$
γ	= ratio of specific heats
ϵ	$= d/l$
ξ	$= \partial H_x/\partial y$
$\xi_{(1)}$	$= \partial h_x/\partial y$
μ	= magnetic permeability
Ω	$= SR_m$
$\varphi, \Phi, \sigma, \Sigma$	= defined by Eq. (24)
ρ	= gas density
σ	= electrical conductivity
η	= entropy

Subscripts and Superscripts

∞	= conditions at infinity
wall	= values at the channel wall
η	= values taken at constant entropy
0	= conditions at $t = 0$
1	= conditions at $M_c = 1$
*	= equilibrium values
$()'$	= derivatives with respect to the coordinate x

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Introduction

IT has been known experimentally, for many years, that variable-area channel flows (Laval nozzles) that decelerate through the sonic point are unstable, in the sense that shock-wave formation is unavoidably associated with such flows. The design of jet engines has always been such that this undesirable phenomenon is made to occur as close to the section of minimum area (the throat where the Mach number is unity) as possible, in order to reduce shock losses.

In light of the current research in the fields of magnetogasdynamic (MGD) power generation and space propulsion by means of ionized gas flow in channels, it is of interest to consider the propagation of weak but finite-amplitude wave motions in MGD channels in order to determine their asymptotic behavior. It seems important to determine whether decelerating MGD channels, such as are used to generate electric power, are necessarily unstable in the same sense, and if so at what flow speed, or whether there are possibilities of suppressing shock-wave formation. It is also of interest to see whether such instability may occur in accelerating MGD channels.

The theory of nonlinear wave motion in a uniform gasdynamic medium was studied for the first time by Riemann[1] in 1859. The concept of Riemann invariants in the theory of characteristics has subsequently been used with much success in various contexts.

The study of finite-amplitude waves that propagate with the velocity of sound relative to some nonuniform fluid motion[2-4] has led to a mathematical explanation of the stability of such wave motions in both accelerating and decelerating nonmagnetic channel flows. In particular, the formation of shock waves in diffusers that decelerate through the sonic point was predicted.

As the nonlinear steepening[1] was shown to be a function of the specific type of initial wave motion (compression pulse or expansion pulse), the asymptotic behavior of waves in nonuniform (accelerating or decelerating) flows was shown to be a function of the nonuniformity[2–4] as well as the nature of the initial disturbance.

The interaction of a magnetic field and a velocity field leads to the creation of an additional body force, the Lorentz force, when the fluid is a conducting medium, e.g., an ionized gas. The effect of this force on the steady-state behavior of one-dimensional channel flows (acceleration, deceleration, choking) has been discussed in detail.[5] The fact that constant-area acceleration is made possible by this interaction and its effect on the velocity field in general lead to the belief that the stability of wave motions in MGD channels will be grossly affected by this new magnetogasdynamic force.

The purposes of this paper are to examine the effect of a magnetic field on the quasi-one-dimensional assumption of ordinary gasdynamics, to consider the asymptotic behavior of finite-amplitude wave motions, and to predict whether shocks will form and what type of shocks, since there are many possibly magnetogasdynamic discontinuities. The possible elimination of shock waves by appropriately placed magnetic-field distributions would, of course, lead to higher efficiencies in the extraction of electrical power from supersonic ionized gas flows and similarly reduce losses in propulsive systems where shocks were previously unavoidable.

It will be assumed here that the electrical conductivity is a scalar; i.e., Hall currents are neglected. Displacement currents, heat conduction, and viscous effects are also neglected. The magnitudes of the magnetic Reynolds number, i.e., motion-induced effects, and of the interaction parameter, i.e., the effect of the magnetic field on the fluid, are prescribed by the range of validity of the one-dimensional approximation.

Formulation

1. Governing Equations

When a magnetic field is applied transverse to the direction of an undisturbed channel flow (Fig. 1), there is a tendency for the magnetic lines of force to be stretched in the direction of the equilibrium flow, because of the interaction between the magnetic and velocity fields. This interaction is measured in part by the magnetic Reynolds number R_m, where $R_m = 4\pi\sigma U_\infty \mu d$ and d is a characteristic length, which in this case is the channel height at entrance, or by the regulation of the externally applied electric and magnetic fields; e.g., this is a necessary condition when the fluid is perfectly conducting.

We consider here a compressible, two-dimensional flow of a conducting fluid in a channel of variable cross-sectional area and finite length. We will allow only those channels for which the height-to-length parameter is small. In other words, the length that is characteristic of changes along the channel (l) is much greater than the length characteristic of changes across the channel (d), so that although variations along the channel are not zero, they are still small compared to variations across the channel. We shall also assume that the cross-sectional area varies slowly with the coordinate measured along the flow direction; this is a necessary condition for any one-dimensional channel-flow approximation.

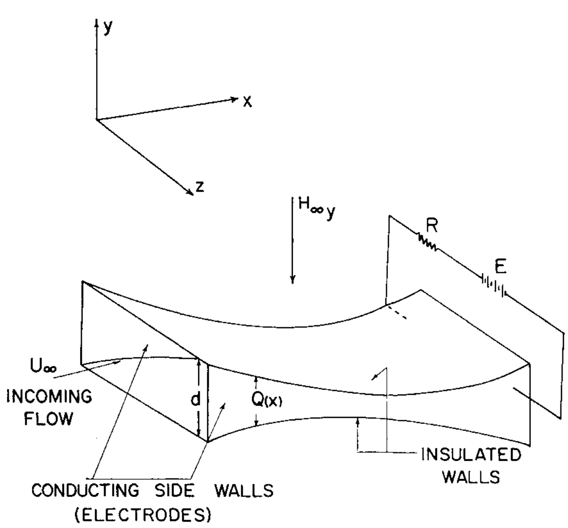

Fig. 1 Channel flow geometry.

The general two-dimensional equations are

Momentum

$$\rho(D\mathbf{q}/Dt) + \nabla p = \mathbf{j} \times \mathbf{B} \tag{1}$$

Mass Continuity

$$(\partial\rho/\partial t) + \nabla \cdot \rho\mathbf{q} = 0 \tag{2}$$

Entropy

$$\rho T(D\eta/Dt) = j^2/\sigma \tag{3}$$

Ampère's Law

$$\nabla \times \mathbf{h} = 4\pi\mathbf{j} \qquad \nabla \times \mathbf{H}_\infty = 0 \tag{4}$$

Faraday's Law

$$\nabla \times \mathbf{E} = -\mu(\partial\mathbf{h}/\partial t) \tag{5}$$

Ohm's Law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{q} \times \mathbf{B}) \tag{6}$$

State

$$p = \rho RT \tag{7}$$

Two-Dimensionality

$$\partial/\partial z \equiv 0 \qquad \mathbf{q} = (u, v, 0) \tag{8}$$

The last statement means that electrode boundary layers are neglected and the flow properties are assumed uniform in the (z) direction.

This set of equations can be reduced by stipulating the aforementioned conditions on the channel-area variation and the ratio of characteristic lengths. The important nondimensional parameters that must be specified in this reduction process are the magnetic Reynolds number and so-called "interaction parameter" (S). By a very simple order-of-magnitude analysis, it is possible to show that for the crossed-fields channel a set of quasi-one-dimensional equations, very similar to those equations applicable in ordinary gasdynamic channels, can be derived.[6] Although it is generally true that the transverse velocity component v and the transverse pressure gradient are small in ordinary channel flows (of the type being considered), it is possible that the transverse magnetic body force will magnify these perturbations. The magnetic Reynolds number, interaction parameter, and applied electric field, as well as geometry, have a large influence on this effect.

The usual averaging procedure of integrating across the channel and replacing all flow variables by their average values leads to the one-dimensional system, as the questionable pressure gradient has zero for its average value. It is interesting that, in the longitudinal momentum equation, the Lorentz-force contribution remains completely unchanged, and that after integration the current density becomes

$$4\pi j = (\partial h_x/\partial y) - (\partial h_y/\partial x) \tag{9}$$

where h_y has been replaced by its average value and

$$\partial h_x/\partial y = (h_x)_{\text{wall}}/Q(x)$$

This is not surprising, since we know[7] the major contribution to the current density in the equilibrium flow is represented by the term $\partial h_x/\partial y$ when the channels are of the type considered here. In other words, for arbitrary R_m, $4\pi j \sim \partial h_x/\partial y$ or $\partial h_x/\partial y \gg \partial h_y/\partial x$.[7] Some authors appear to have the misconception that "one-dimensional" in the channel flow equations, implies no transverse gradients. This is only true in what is termed "strictly one-dimensional" flow, or motion for which there are no restrictions or boundaries in the transverse or (y) direction. The channel-flow equations are "quasi-one-dimensional" and it is only an approximation or averaging process that eliminates transverse gradients in ordinary channel flows. For the magnetogasdynamic case, as we see, this most certainly does not imply that all transverse gradients are zero.

Although the averaging process is certainly a valid one, it appears desirable to obtain a more exact derivation of the one-dimensional equations, i.e., to determine the orders of magnitude of the nonuniformities and particularly those conditions for which the transverse body force is unimportant. This is done rigorously in Ref. 6 [with attention to the paper of McCune and Sears[7] on the relative magnitudes of the two terms on the right hand side of Eq. (9)]. The resulting equations governing the flow (to order $\epsilon = d/l$) are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\mu H_y}{4\pi\rho}\left(\xi - \frac{\partial H_y}{\partial x}\right) \tag{10}$$

$$\frac{\partial \ln\rho}{\partial t} + u\frac{\partial \ln\rho}{\partial x} + \frac{\partial u}{\partial x} = -u\frac{d\ln Q}{\partial x} \tag{11}$$

$$p = \text{const}\rho^{\gamma} \qquad a^2 = \gamma p/\rho \tag{12}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{\mu H_y}{4\pi\rho Q}\right) = 0 \tag{13}$$

$$\xi = \frac{\partial H_x}{\partial y} = \frac{\partial H_{\infty x}}{\partial y} + \frac{\partial h_x}{\partial y} = \frac{(H_x)_{\text{wall}}}{Q(x)} \tag{14}$$

where $H_{\infty x}$ is due to any fringing of the applied magnetic field. If $H_{\infty y} \equiv$ const, then $H_{\infty x} \equiv 0$.

In deriving these equations, it has been postulated that the parameter $\Omega = SR_m$ is of the order of the ratio of characteristic lengths $\epsilon = d/l$ and that the nondimensionalized current density $j/4\pi\sigma U_\infty H_{\infty y}$ is small. The problem can similarly be done for $j/4\pi\sigma U_\infty H_{\infty y}$ of order one[6] with only a minor correction necessary. The latter case is realized when the magnetic Reynolds number is small, whereas the former implies that the channel flow is running very close to the no-load condition; i.e., E_z/UH_y is approximately unity.

2. Characteristics

The general theory of characteristics for a nondissipative magnetogasdynamic medium is described fully by Friedrichs and Kranzer.[8] From Eq. (10–14) we find that the appropriate propagation velocities, relative to the fluid motion, are $c^2 = a^2 + \alpha^2$ and $c^2 = 0$, where a is the ordinary sound speed, $a = [(\partial p/\partial\rho)_\eta]^{1/2}$ and α is the Alfvén velocity, $\alpha = H_y/(4\pi\rho/\mu)^{1/2}$. These "effective sound speeds" are of course in agreement with Ref. (8) for the crossed-fields geometry. Therefore, we have plane wave propagation at absolute velocities $u + c$ and $u - c$. The former are termed advancing waves and the latter receding waves.[3] In diffusers or decelerating channel flows (Fig. 2) both waves will travel into the channel if there occurs some initial disturbance at the entrance, whereas only a receding wave will enter the channel following a disturbance at the exit.

In nozzle or accelerating-channel flows (Fig. 2) the only possible wave motion due to any disturbance is the propagation of an advancing wave originating at the entrance. There is a basic difference between these two types of wave motion, because although advancing waves have the property that they propagate through the channel and out to infinity, the receding waves asymptotically approach the sonic point, which is not necessarily the geometric throat as it is in ordinary gas-

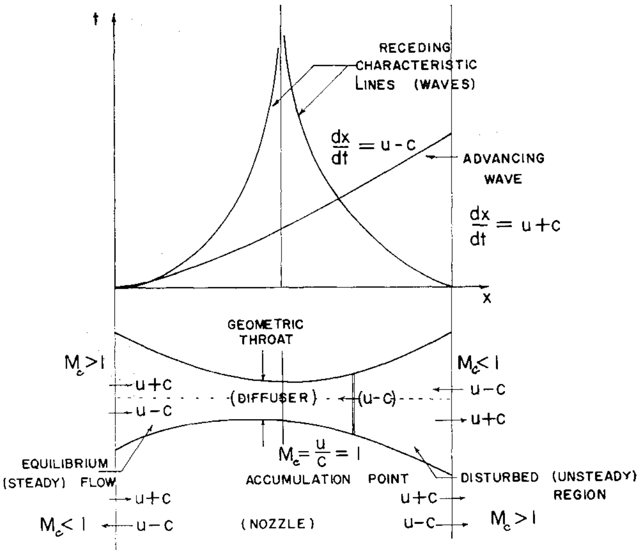

Fig. 2 Wave patterns for both decelerating and accelerating flows.

dynamics. The movement of the sonic point is due to the induced magnetic field or current and is very significant in the subsequent stability considerations. The growth or decay of receding waves is therefore confined to the channel (finite in length), whereas an infinite channel is postulated for the advancing waves.

The compatibility relations for the set of Eqs. (10–14) are

$$\frac{c}{2}\frac{D_\pm \ln\rho}{Dt} \pm \frac{1}{2}\frac{D_\pm u}{Dt} = -\frac{cuA}{2} \pm \frac{\mu H_y}{8\pi\rho}\xi \mp \frac{\alpha^2 A}{2} \tag{15}$$

where

$$A = \frac{d\ln Q}{dx} \qquad \frac{D_\pm}{Dt} = \frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}$$

The positive "characteristic" derivative D_+/Dt corresponds to a derivative along an advancing wave or across a receding wave, whereas the negative derivative D_-/Dt is along a receding and across an advancing wave.

The subsequent analysis is a simple extension of a procedure that was devised in Ref. 3 for the ordinary gasdynamic channel flows. It is a perturbation technique, where a small disturbance is superimposed on a steady flow and its subsequent behavior is traced via the compatibility relations. This is much the same as the Riemann problem,[1] where a small unsteady perturbation is superposed on a steady uniform flow and leads to wave steepening if the initial pulse is of a compressive nature. Let us assume that, because of some disturbance at the exit (or entrance) of the channel, say a local pressure fluctuation, an unsteady wave motion propagates upstream (downstream) (Fig. 2) in a diffuser with a velocity $dx/dt = u - c$. Behind the wave front there is a very complicated system of unsteady wave motions, because of the reflections of the initial pulse which are caused both by area change and the curvature of the magnetic field. The wave front therefore separates the undisturbed flow from the unsteady region. The behavior of the differential equations are considered then only at the wave front of the initial pulse.

Since we are concerned here only with small perturbations of nonvanishing amplitude, we stipulate, as a first approximation, that at the wave front or bifurcation of the two domains the flow variables (ρ, u, a, etc.) undergo only very small changes, and therefore, the unsteady coefficients in the differential equations can be replaced in favor of their equilibrium values. Similarly, derivatives that propagate with the wave can also be replaced with their steady-state values. On the other hand, normal derivatives are direct functions of the initial disturbance and are not necessarily continuous across the wave front; in fact, the jump in normal derivative is some finite quantity and consequently these cannot be replaced by their equilibrium values. In terms of the characteristic theory, we are simply employing the definition that characteristic lines are lines across which normal derivatives, but not the variables themselves, may be discontinuous. Since the particle acceleration is prescribed by such normal derivatives, and since the strength of the initial disturbance is merely a function of the jump in acceleration across the wave front, this quantity turns out to be a convenient stability parameter; i.e., the growth or decay of the acceleration discontinuity implies the same for the disturbance and hence the ultimate stability of the steady channel flow to such a disturbance. A series of papers[9–11] have been written which apparently utilize this procedure; however, as explained in Ref. 6, the author appears to have made some fundamental errors.

The equations that describe the equilibrium channel flow are

$$UU' + c^{*2} R'/R = (\mu H_y^*/4\pi R)\xi^* - \alpha^{*2}A \tag{16}$$

$$U'/U + R'/R = -A \tag{17}$$

$$\xi^* = (H_x^*)_{\text{wall}}/Q_{(x)} = \xi_{(1)}^* + (H_{y_\infty}^*)' \tag{18}$$

where the primes denote differentiation with respect to the axial coordinate (x), and the capital letters or stars denote undisturbed values. By some very obvious interchanges we find the following two relationships:

$$\frac{U'}{U}(M_c^{*2} - 1) = (1 - \beta^{*2})\left(\frac{\xi^*}{H_y^*}\right) + \beta^{*2}A \tag{19}$$

$$\frac{U'}{U}(M_a^{*2} - 1) = \frac{1 - \beta^{*2}}{\beta^{*2}}\left(\frac{\xi_{(1)}^*}{H_y^*}\right) + A \tag{20}$$

Where M_a is the usual Mach number, $M_a = u/a$, and $\beta^{-2} = (c/a)^2 = 1 + (\alpha/a)^2$,

$$\xi_{(1)}^* = (h_x^*)_{\text{wall}}/Q(x) = \sigma(E^* - UH_y^*)$$

From the definitions of the Alfvén velocity and ordinary sound velocity the following conditions are obtained:

$$(a^*)' = -[(\gamma - 1)/2a^*][UU' - (1 - \beta^{*2})(c^{*2}\xi^*/H_y^*)] \tag{21}$$

$$(\alpha^*)' = \alpha^*/2[A - (U'/U)] \tag{22}$$

The induced current $\xi_{(1)}^*$ affects the flow in a manner very similar to the area change [see Eqs. (19) and (20)]. Because of this phenomenon it is possible to accelerate or decelerate a constant-area channel flow, i.e., propulsion ($\xi_{(1)}^* > 0$) or power generation ($\xi_{(1)}^* < 0$). Whether such flows are accelerating or decelerating depends on the entrance Mach numbers, i.e., subsonic $M_a < 1$ or supersonic $M_a > 1$. From (19) and (20) we see immediately that neither of the sonic points $M_a = 1$ or $M_c = 1$ necessarily occurs at a geometric throat. For $\xi^* < 0$, it is found in the diverging section of the channel, and for $\xi^* > 0$, it is found in the converging section of the channel.

For this characteristic theory we have noted that normal derivatives are necessarily discontinuous across the wave front and that some combination of these derivatives determines the asymptotic stability of the initial perturbations. At first glance we might suspect that the stability parameter is a direct analog of the parameter applicable in ordinary channel flows,[3] which is related directly to the so-called Riemann invariants $P^* = a/(\gamma - 1) + u/2$, $Q^* = a/(\gamma - 1) - u/2$. In the compatibility relations (15), expressions similar to $a(\partial \ln\rho/\partial x) + \partial u/\partial x$ can be replaced by $\partial[a/(\gamma - 1) + u/2]/\partial x = \partial P^*/\partial x$ by using the isentropic relationship. The acceleration jump is now directly related to $\partial P^*/\partial x$ and $\partial Q^*/\partial x$ and a great deal of simplicity results. In the case considered here, we have terms similar to $c(\partial \ln\rho/\partial x) + \partial u/\partial x$, and although a is only a function of the density, c is a function of the density and area variation,

$$c^2 = a^2 + \alpha^2 = a^2(\rho) + \alpha^2(\rho, Q) = a^2(\rho) + \alpha^2(\rho, \xi)$$

and a direct analogy is possible only for a constant-area duct.

Secondly, it has been argued that the flow variables change very little across the wave front and that we can replace the unsteady coefficients in the differential equations by their equilibrium values. Since we have a magnetic field present, it is necessary to be a little more careful. It is possible that we have a weak wave in so far as the flow variables are concerned but that large discontinuities in the magnetic field are allowed. This might be the case if the wave were a current sheet. From the continuity of the magnetic field we are immediately assured of the continuity of the normal component of the magnetic field h_x regardless of the wave strength, implying that the unsteady (perturbation) current density j' is given as $4\pi j' = \partial h_y/\partial x$, whereas the steady-state current density is, as previously stated, approximated by $4\pi j = \partial h_x/\partial y$. Since the fluid is not perfectly conducting, and since it has been shown that the current density (both steady-state and perturbed components) is small,[6] we can ignore the possibility of a current sheet. From Ampére's law we are guaranteed that there is no discontinuity in the transverse magnetic field component h_y; however, $\partial h_y/\partial x \sim j$ is discontinuous as we would expect.

The particle acceleration in the perturbed flow is Du/Dt and can be directly related to the normal (in this case positive) derivative of the velocity across the wave front. The equilibrium acceleration is $Du/Dt = UU'$ and consequently the acceleration jump is given by

$$(Du/Dt) - (DU/Dt) = -s \tag{23}$$

where s can now be termed a stability parameter. If s tends asymptotically to zero, the normal derivatives become continuous and the wave has dissipated, whereas if s tends to infinity, the derivatives grow; the assumed flow breaks down and shock-wave formation is predicted.

It is convenient to define the following terms:

$$\varphi = \frac{c}{2}\frac{D_+ \ln\rho}{Dt} \qquad \sigma = \frac{1}{2}\frac{D_+ u}{Dt}$$

$$\Phi = \varphi^* = \frac{c^*}{2}(U + c^*)\frac{R'}{R} \qquad \Sigma = \sigma^* = \frac{U + c^*}{2}U' \tag{24}$$

The compatibility relations then can be written as

$$\varphi + \sigma = \frac{-cuA}{2} + \frac{\mu H_y}{8\pi\rho}\xi - \frac{\alpha^2 A}{2} \tag{25}$$

$$I = \frac{c}{2}\frac{D_- \ln\rho}{Dt} - \frac{1}{2}\frac{D_- u}{Dt} = -\frac{cuA}{2} - \frac{\mu H_y}{8\pi\rho}\xi + \frac{\alpha^2 A}{2} \tag{26}$$

where the letter I is used simply for convenience in the later calculations. However, in view of the continuity assumption we obtain the following relation, which reduces the problem to that of one unknown instead of the original two, ϕ and σ. Along the wave front

$$[\varphi - \Phi] + [\sigma - \Sigma] = 0 \tag{27}$$

The acceleration discontinuity can now be expressed as a function of these terms. Along the characteristic wave being considered,

$$\frac{Du}{Dt} - \frac{DU}{Dt} = -s = (\sigma - \Sigma) = -(\varphi - \Phi) \tag{28}$$

3. Differential Equation for s

The parameter s describes the asymptotic behavior of the wave motion; consequently, it is desirable to find some differential equation that governs the behavior of s as the pulse propagates through the channel. In particular for the receding waves considered here, we are interested in its growth near the accumulation or sonic point, whereas for the advancing waves, its growth at infinity is pertinent. Let us differentiate the compatibility Eq. (27) along a positive characteristic, i.e.,

$$\frac{D_+}{Dt} I = \frac{D_+}{Dt}\left[\frac{c}{2}\frac{D_- \ln\rho}{Dt} - \frac{1}{2}\frac{D_- u}{Dt}\right] = -\frac{A}{2}\left[\frac{D_+(uc)}{Dt} - \frac{D_+\alpha^2}{Dt}\right] + Z(c, \rho, u, Q, \xi)$$

Also,

$$\frac{D_-(\varphi - \sigma)}{Dt} - \frac{D_+ I}{Dt} = \left(\frac{c}{2}\frac{\partial \ln\rho}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial x}\right)\left[\frac{D_-}{Dt}(u + c) - \frac{D_+}{Dt}(u - c)\right] + \frac{1}{2}\left(\frac{D_+ \ln\rho}{Dt}\frac{D_- c}{Dt} - \frac{D_+ c}{Dt}\frac{D_- \ln\rho}{Dt}\right)$$

The addition of these two equations yields

$$\frac{D_-(\varphi - \sigma)}{Dt} = -\frac{A}{2}\left[\frac{D_+(uc)}{Dt} - \frac{D_+\alpha^2}{Dt}\right] + \frac{1}{2}\left(\frac{D_+ \ln\rho}{Dt}\frac{D_- c}{Dt} - \frac{D_+ c}{Dt}\frac{D_- \ln\rho}{Dt}\right) + \frac{1}{4}\left[\left(\frac{D_+}{Dt} - \frac{D_-}{Dt}\right)\ln\rho - \frac{1}{c}\left(\frac{D_+}{Dt} - \frac{D_-}{Dt}\right)u\right] \times \left[\frac{D_- u}{Dt} + \frac{D_- c}{Dt} - \frac{D_+ u}{Dt} + \frac{D_+ c}{Dt}\right] + Z'(\rho, u, c, Q, \xi) \quad (29)$$

The functions Z and Z' contain only those quantities that are continuous across the wave front and, as we shall see, they need not be known explicitly. On the steady side of the wave front this equation must also apply with Φ replacing φ and Σ replacing σ. If the latter is subtracted from the perturbed Eq. (29), with all coefficients replaced by their steady counterparts, and the relationship between φ and σ being utilized (27), we obtain the following differential equation for s†:

$$\frac{D_-}{Dt} s = \frac{1}{2c^*}(J_1 s^2 + J_2 s) \quad (30)$$

Where

$$J_1 = 3 + (\gamma - 2)\beta^{*2}$$

$$J_2 = Ac^{*2}/2[3M_c^* - 1 - (\gamma - 2)\beta^{*2} - 3M_c^*\gamma\beta^{*2}] - c^*U'/2[3 + (1/M_c^*)][3 + (\gamma - 2)\beta^{*2}] = C^{*2}A(\beta^{*2} - 1) - C^{*2}(M_c^*)'[3 + (1/M_c^*)] \quad (31)$$

The final differential Eq. (30) is a simple nonlinear equation for the parameter s. The simplicity is due to the continuity assumption, which causes the coefficients J_1 and J_2 to be functions of the equilibrium flow alone. The coefficient J_2 is a function of the channel flow, geometry, and magnetic field. For a uniform steady flow $J_2 \equiv 0$ and what remains is a magneto-hydrodynamic analog of Riemann steepening. Note that J_1 is only slightly affected by the magnetic field, viz., $\gamma + 1 \leqslant J_1 \leqslant 3$, $0 < \beta^* < 1$ and hence the steepening phenomenon is relatively unchanged.

Solution

The problem has now been reduced to a solution of the nonlinear stability of the singular point $s = 0$. J_1 and J_2 are prescribed by the known steady channel flow, but the explicit dependence on the streamwise coordinate (x) is not given. Since the waves tend to accumulate at the sonic point and reach this region in a relatively short time, we shall consider the behavior of the differential equation, as a first approximation, in the neighborhood of the accumulation point, i.e., we replace J_1 and J_2 by their sonic values. The equation is now of the type $dx/dt = Ax^2 + Bx$, where A and B are constants, and from Liapunov's first stability criterion, a sufficient condition for the asymptotic stability (decay) of the nonlinear equation is the asymptotic stability of the linear approximation,[12] $dx/dt = Bx$; i.e., the necessary stability condition is

$$B = J_2(M_c^* = 1)/2c^* < 0 \quad (32)$$

This is obviously impossible for an ordinary decelerating nonmagnetic channel flow [see Eqs. (31) and (32)] where $\beta^2 = 1$, but possible for the MGD case. However, this criterion provides only sufficient and not necessary conditions for the stability, and therefore we cannot deduce anything concerning the instability of the nonlinear equation from the linear approximation. We shall subsequently show that these are also necessary conditions as well. The sufficiency condition could have been obtained directly for all x by noting that the function $-J_2 s$ is a Liapunov function for the system providing that $J_2 < 0$.[12]

Up to this point we have only been able to obtain approximate, sufficient stability conditions, which might in fact be too restrictive. To obtain further information we take a slightly different approach and make the substitution $z = 1/s$. The governing differential equation that was nonlinear in s is linear in z and is identical to the equation obtained by Meyer, although the coefficients are, of course, different:

$$(D_-/Dt)z = -1/2c^*(J_1 + 2J_2) \quad (33)$$

The solution can be written in integral form

$$\left(\frac{L}{L_0}\right) z = z_0 - \frac{1}{2}\int_0^t J_1 \frac{L}{L_0}\frac{dt}{c^*} \quad (34)$$

where

$$\frac{L}{L_0} = \exp\frac{1}{2}\int_0^t \frac{J_2}{c^*}\, dt \quad (35)$$

and $dx/dt = U - c^*$ at the wave front, and at $t = 0$, $x = x_0$, $z = z_0$. The constant $z_0 = 1/s_0$, which depicts the initial displacement from equilibrium, is in general a large finite quantity, since we are only interested in relatively small departures from equilibrium. This implies that the initial acceleration jump, which is inversely proportional to z_0, is small and that the pulse does not have a large entropy change associated with it as a shock wave might have. We have postulated that this departure from equilibrium is in fact an unsteady perturbation created by some local pressure change. The relationship between s or z and this local unsteady pressure fluctuation is $(1/\rho c)(\partial p/\partial t) = \beta^2 s(1 - M_c) = \beta^2(1 - M_c)/z$. The formula, which is valid at the wave front for the receding waves, is a direct result of Eqs. (10, 11, 19, 23, 24, and 27). For a receding wave originating at the exit (or entrance), compression pulses correspond to positive values of s or an increase in the acceleration from the equilibrium flow to the perturbed flow, while expansion waves correspond to negative values of s. It is interesting to note that, for some given initial discontinuity s_0, the required pressure change decreases as the strength of the magnetic field increases (β decreases).

In the following discussion, we will postulate that the initial value of the disturbance s_0 is an arbitrarily small but nonzero quantity. This assumption was implicit when the stability of the singular point $s = 0$ in the nonlinear equation was discussed, for it might be possible to pick an initial value so large (or so far from the singular point) that the solution would

† See Ref. (6) for a complete derivation.

become unstable. In terms of the wave analysis, this would mean an upper bound on the strength of the initial disturbance. Table 1 will be useful and is merely stated here for future reference.

As mentioned previously, the magnetic field, which has only a slight effect on J_1, has a marked influence on the coefficient J_2. In ordinary gasdynamic channel flows $J_2(x)$ is dependent on only one of the variables M_x, U, or A; in magnetogasdynamic channels, it is a function of any two of the variables M_a, M_c, U, A, or $\xi_{(1)}$. Also of importance is the sign of $J_2(x)$, which for the case of zero magnetic field is a direct function of the sign of M_a'. Under the influence of a magnetic field both the sign and the behavior of J_2 can be altered. For a given Mach-number distribution we can prescribe $J_2(x)$ arbitrarily and in turn determine the geometry and necessary magnetic field distribution or more specifically the current density as prescribed by Ohm's law. Although the function $J_2(x)$ can assume almost any variation, we will consider two distinct cases: the first in which it is positive and the second in which it is negative or tends to zero negatively and slowly enough. Since the asymptotic behavior of the wave is prescribed mainly by the equilibrium flow in some neighborhood of the sonic point and since the growth of the wave from the point of inception to this neighborhood will be small for an arbitrarily small initial disturbance, it is really necessary that these conditions be satisfied only near the sonic point.

Proposition 1: $J_2 > 0$, Primarily as $t \to \infty$ or in the Vicinity of the Sonic Point

This prescribes a channel flow which is very similar to an ordinary decelerating channel flow for which $M_a' < 0$ and $\beta = 1$, and therefore we would expect that the behavior of the MGD waves will also be similar.

Since $J_2 > 0$ it is obvious that $L/L_0 \to \infty$ as $t \to \infty$, and as depicted in Table 1, B4, $s \to \infty$ and an instability is predicted for the propagation of a compression pulse. On the other hand the case for the expansion wave is not as clear [see (A3)], because

$$\lim_{t\to\infty} z = \lim_{t\to\infty} (z_0 - \Delta)/(L/L_0)$$

where both $\Delta \to \infty$ and $L/L_0 \to \infty$ as $t \to \infty$. Since the exact time (or x) dependence is not known, we appeal to the form of the differential equation (33),[3] because if $z < -J_1/J_2$, then $D_-z/Dt > 0$, whereas if $z > -J_1/J_2$, then $D_-z/Dt < 0$. Therefore, for any initial value z_0, z must asymptotically approach $-J_1/J_2$ as a limiting value. Therefore, at the sonic point we find that

$$\frac{D_u}{Dt} \to UU'(-1) + (UU' - U^2\xi_1^*) \times \left\{\frac{(1-\beta^{*2})[1-\beta^{*2}(2\gamma-1)]}{[3+(\gamma-2)\beta^{*2}]\beta^{*2}}\right\} \quad (36)$$

the expansion pulse neither grows or dissipates, but attains some new equilibrium value that is greatly affected by the magnetic field. We see that, when $\beta^2 = 1^{(3)}$ or $\beta^2 = 1/(2\gamma - 1)$, the unsteady acceleration approaches, as its equilibrium value, the negative of the steady-state value or the value appropriate to an accelerating channel. The equilibrium value varies considerably with other values of β and the sign of $\xi_{(1)}$, i.e., whether it is acting as a generator or an accelerator.

Table 1 Asymptotic behavior of stability parameter

A) For an expansion wave ($z_0 < 0$), as $t \to \infty$
1) If $L/L_0 \to 0$ then $z \to -\infty$ and $s \to 0$
2) If $0 < L/L_0 < \infty$ then $z \to -\infty$ and $s \to 0$
3) If $L/L_0 \to \infty$ then the integral solution provides an indeterminate result for z
B) For a compression wave ($z_0 > 0$), as $t \to \infty$
1) If $L/L_0 = o(t^{-(1+e)})$, where $e > 0$, then the integral solution provides an indeterminate result for z
2) If $L/L_0 = o(t)$, then z passes through zero at some time $t \leqslant \infty$
3) If $0 < L/L_0 < \infty$, then z passes through zero at some time $t \leqslant \infty$
4) If $L/L_0 \to \infty$, then z passes through zero at some time $t \leqslant \infty$

Proposition 2: $J_2 < 0$ or $(J_2)^{-1} = o(-t/(1+e))$ as $t \to \infty$, where $e > 0$

This is the more interesting of the two categories since it cannot occur in ordinary gasdynamic decelerating channel flows. It is possible here only because the applied magnetic field can effectively move the sonic transition point into the diverging section of the channel. For expansion waves, case A1 of Table 1 is appropriate, and therefore, such waves will attenuate, so that the equilibrium channel flow is "stable‡" to these disturbances. If there are conditions for which compression pulses also attenuate, shockless decelerating channel flows with applied magnetic fields are possible. Based on the conditions that define proposition 2, we find immediately that

$$L/L_0 = 0(t^{-(1+e)}) \quad \text{or} \quad L/L_0 \to 0 \quad \text{as} \quad t \to \infty$$

and

$$\Delta = \int_0^t \frac{J_1}{2c}\left(\exp\int_0^{t'} \frac{J_2}{2c}\,dt''\right)dt' \to \Delta_0 < \infty \quad (37)$$

since

$$\lim_{t\to\infty} z \to (z_0 - \Delta)/(L/L_0)$$

the wave will attenuate providing

$$z_0 > \Delta_0 \quad \text{or} \quad s_0 < (\Delta_0)^{-1} \quad (38)$$

In other words, the strength of the initial perturbation must not exceed some prescribed value, which is a direct function of the strength of the magnetic field, because a larger magnetic field implies a larger negative value of J_2 and in turn a smaller Δ_0 [see Eq. (31)]. Hence, by increasing the magnetic field, the upper bound for initial disturbances that will decay and render the flow "stable" is also increased. Since only arbitrarily small perturbations are being considered here, we postulate that large departures from equilibrium would not be contained in this classification and might imply that a shock-wave or entropy discontinuity was being introduced into the channel. Therefore, MGD decelerating channel flows are "stable" to all disturbances (receding wave motions) providing the conditions on Proposition 2 are satisfied. The neutral-stability condition is then $J_2 = 0$, or

$$(Q'/Q)(\beta^{*2} - 1) - (M_c^{*\prime})(3 + 1/M_c^*) = 0 \quad (39)$$

where $\beta^{*2} = 1 + KQ^2R^{(\gamma-2)}$, $K = \text{const}$

For the special case of $\gamma = 2$ the condition for neutral stability can be integrated in closed form, and we find that

$$(1 - \beta_1^2)(Q/Q_1)^2 = e^{-6(M-1)}/M_c^2 - \beta_1^2 \quad (40)$$

where all quantities are equilibrium values (stars have been dropped). Subscript one denotes values where M_c is unity. This equation is a very sensitive function of the parameter β_1 as illustrated in Fig. 3.

Figure 3 defines the Mach-number distribution that is required in order to insure a stable decelerating channel flow. For a given value of β_1, the graph depicts lines of neutral stability, and any equilibrium Mach-number distribution above this line, i.e., lower Mach-number gradient, implies

‡ The word "stable" is used loosely to define flows for which instability is not predicted, since all wave motions attenuate; hence, the possibility of a stable condition exists.

attenuation of all disturbances, whereas steeper distributions lead to an instability. As the magnetic field strength is increased, the region for which a "stable" flow exists is also increased. However, the biggest changes occur significantly the instant any magnetic field is applied, because in the ordinary nonmagnetic channel flows, there is no decelerating flow that is "stable," whereas for $\beta_1 = 0.99$ (approximately 200 gauss at a gas pressure of one atmosphere!), a large class of distributions is already possible. As the strength gets very large, a limit line is approached [see Eq. (40) as $\beta_1 \to 0$].

Up to this point, we have only considered the motion of the receding waves. We have already explained that, unlike this type of wave motion, which leads to accumulation within the channel, advancing waves propagate through any finite-length channel, and although shock-wave formation might be predicted, only by considering an infinitely long channel are we guaranteed that the pulse has not passed out of the exit before a shock wave has formed. The analyses are identical, with the exception that infinite time is now comparable to infinity in the downstream direction ($x \to \infty$), and is carried out in detail in Ref. (6). The result of this analysis is the calculation of the appropriate downstream channel-area variation, which insures attenuation of all wave motions. In one particular case where the steady-state current density is assumed to be zero beyond some station in the channel we find the following asymptotic conditions on the channel area Q:

$$\left|Qx^{-\{4/[1+1/\beta_\infty^2+(1-\beta_\infty^2)(5-\gamma)]\}}\right| \to \infty \quad \text{as} \quad x \to \infty \qquad (41)$$

When the magnetic field is zero, we recover Meyer's condition[3]

$$|Qx^{-2}| \to \infty \quad \text{as} \quad x \to \infty$$

While for $\beta_\infty^2 = \frac{1}{2}$, we find that for $\gamma = 2$

$$|Q_x^{-8/9}| \to \infty \quad \text{as} \quad x \to \infty \qquad (42)$$

In fact, as β_∞ approaches zero, the required geometry at infinity approaches a constant-area channel. Several other cases are treated in Ref. (6).

Conclusion

It has been shown that the application of a perpendicular§ magnetic field can in certain instances have a stabilizing effect on decelerating channel flows. Although it has not been shown explicitly, it is also true that the magnetic field can have a destabilizing effect as well. Most significant is the result that even relatively small applied magnetic fields yield large areas where all wave motions attenuate. The instabilities that are predicted for ordinary gasdynamic channel flows are no longer a result of the theory, and the possibility of a stable decelerating supersonic flow exists. Finally, the size of the domains for which all perturbations decay increases as the strength of the magnetic field increases.

§ For the aligned-fields case see Ref. 6.

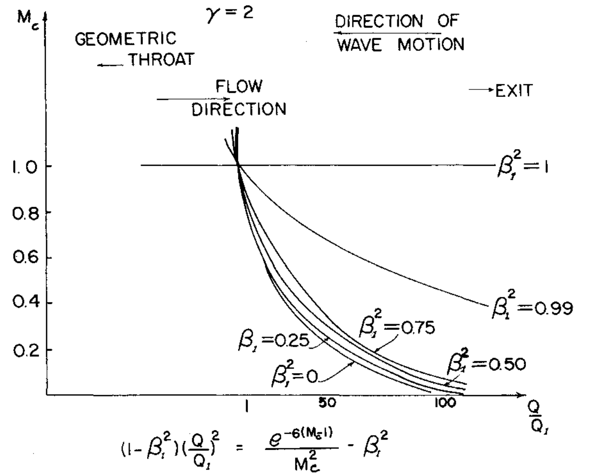

Fig. 3 Neutral stability curves for $\gamma = 2$ and several values of β.

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